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ON DOUBLY STOCHASTIC MATRICES WITH PRESCRIBED ELEMENTARY DIVISORS

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Abstract

In this paper, we show how to construct an $n \times n$ doubly stochastic matrix, $n \geq 4$, with spectrum $\{1, -\frac{1}{n-1}, \cdots, -\frac{1}{n-1}\}$ and with prescribed elementary divisors of the form $\left(\lambda + \frac{1}{n-1}\right)^k$, $k \geq 2$. This construction gives an answer to a question stated by Minc in [3].

1. Introduction

Let $A \in \mathbb{C}^{n \times n}$ and let

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$$J(A) = S^{-1}AS = \begin{bmatrix} J_{n_1(\lambda_1)} & & & \\ & J_{n_2(\lambda_2)} & & \\ & & \ddots & \\ & & & J_{n_k(\lambda_k)} \end{bmatrix}$$

be the *Jordan canonical form* of A (hereafter *JCF* of A). The $n_i \times n_i$ submatrices

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad i = 1, 2, \dots, k$$

are called the *Jordan blocks* of $J(A)$. Then, the *elementary divisors* of A are the polynomials $(\lambda - \lambda_i)^{n_i}$, that is, the characteristic polynomials of $J_{n_i}(\lambda_i)$, $i = 1, \dots, k$.

A matrix $A = (a_{ij})_{i,j=1}^{n}$ is called *quasi-stochastic*, if all its row sums are 1 and it is called *doubly quasi-stochastic*, if all its row sums and column sums are 1. It is clear that any quasi-stochastic (doubly quasi-stochastic) matrix has the eigenvector $\mathbf{e} = (1, 1, \dots, 1)^T$ corresponding to the eigenvalue 1. Denote by $\mathbf{e}_k$ the vector with one in the k-th position and zeros elsewhere. A nonnegative quasi-stochastic matrix is called *stochastic*, while a nonnegative doubly quasi-stochastic matrix is called *doubly stochastic*. In other words, a nonnegative matrix A is stochastic, if and only if $A\mathbf{e} = \mathbf{e}$, and it is doubly stochastic, if and only if $AJ_n = J_nA = J_n$, where J_n is the $n \times n$ matrix, whose entries are all $\frac{1}{n}$. We shall denote by E_{ij} the $n \times n$ matrix with 1 in the (i, j) position and 0's elsewhere and by $\operatorname{rank}(X)$ the rank of the matrix X.

In [1, Theorem 2.8], the authors show that if $\Lambda = \{1, \lambda_2, \dots, \lambda_n\}$ is a list of real numbers satisfying certain conditions, then there exists a positive doubly stochastic symmetric matrix with spectrum Λ and with arbitrarily prescribed elementary divisors. It is also proved in [1] that Theorem 2.8 contains Theorem 2 in [2] and that $\Lambda = \{1, \alpha, \dots, \alpha\}$, $\alpha \in \mathbb{R}$, is the spectrum of an $n \times n$ positive symmetric doubly stochastic matrix, if and only if $-\frac{1}{n-1} < \alpha < 1$.

In [2], Minc sets the question: Given a doubly stochastic matrix A, does there exist doubly stochastic matrices with the same spectrum as A and any legitimately prescribed elementary divisors ? (that is, its product has to be equal to the characteristic polynomial of A and they cannot include $(\lambda - 1)^k$ with $k > 1$). Although Minc gives a positive answer to this question when A is a positive diagonalizable doubly stochastic matrix, he also shows that, in general, the answer is negative [2, Theorem 3]: The matrix

$$\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

has eigenvalues $1, -\frac{1}{2}, -\frac{1}{2}$, but no 3×3 doubly stochastic matrix has elementary divisors $\lambda - 1$ and $\left(\lambda + \frac{1}{2}\right)^2$. Minc also shows that this result does not extend to $n = 4$: The doubly stochastic matrix

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}$$

has eigenvalues $1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}$ and elementary divisors $\lambda - 1$, $\left(\lambda + \frac{1}{3}\right)^2$, and $\lambda + \frac{1}{3}$. Minc says that it is not known what the answer is for larger n. In this paper, we partially answer the question for $n \geq 4$. In particular, we show how to construct an $n \times n$ doubly stochastic matrix with spectrum $\{1, -\frac{1}{n-1}, \cdots, -\frac{1}{n-1}\}$ and elementary divisors of the form $\left(\lambda + \frac{1}{n-1}\right)^k$, $k \geq 2$. Our results are constructive, in the sense that we can always construct the corresponding doubly stochastic matrix. Examples are given to illustrate this construction.

2. Main Results

We shall need the following lemmas given in [2] and [3]:

Lemma 2.1 [2]. *Let A be an $n \times n$ doubly quasi-stochastic matrix with elementary divisors*

$$(\lambda - 1), (\lambda - \lambda_2)^{n_2}, \ldots, (\lambda - \lambda_k)^{n_k}, \; n_2 + \cdots + n_k = n - 1,$$

and let $\theta \in \mathbb{C}$, $\theta \neq 0$. Then $(1 - \theta)J_n + \theta A$ is a doubly quasi-stochastic matrix with elementary divisors

$$(\lambda - 1), (\lambda - \theta\lambda_2)^{n_2}, \ldots, (\lambda - \theta\lambda_k)^{n_k}, \; n_2 + \cdots + n_k = n - 1.$$

Lemma 2.2 [3]. *Let $\mathbf{q} = (q_1, \ldots, q_n)^T$ be an arbitrary n-dimensional vector and A be a matrix with constant row sums equal to λ_1 and JCF*

$$S^{-1}AS = J(A) = \begin{bmatrix} \lambda_1 & & & \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{bmatrix}.$$

Let $\lambda_1 + \sum_{i=1}^n q_i \neq \lambda_i$, $i = 2, \ldots, n$. Then, the matrix $A + \mathbf{eq}^T$ has JCF $J(A) + (\sum_{i=1}^n q_i)E_{11}$. In particular, if $\sum_{i=1}^n q_i = 0$, then A and $A + \mathbf{eq}^T$ are similar.

We also need the following lemma:

Lemma 2.3. *Let* $k \geq 2$. *Then, the* $k \times k$ *matrix*

$$F = \begin{bmatrix} -1 & \frac{1}{k-1} & \cdots & \frac{1}{k-1} & \frac{1}{k-1} \\ \frac{1}{k-1} & -1 & \ddots & \vdots & \frac{1}{k-1} \\ \vdots & \frac{1}{k-1} & \ddots & \frac{1}{k-1} & \vdots \\ \frac{1}{k-1} & \vdots & \ddots & -1 & \frac{1}{k-1} \\ \frac{1}{k-1} & \frac{1}{k-1} & \cdots & \frac{1}{k-1} & -1 \end{bmatrix} \tag{1}$$

has $\operatorname{rank}(F) = k - 1$.

Proof. Let $\mathbf{q} = (1, -\frac{1}{k-1}, \ldots, -\frac{1}{k-1})^T$ and $\mathbf{e} = (1, \ldots, 1)^T$ be vectors in $\mathbb{R}^k$. Then $\mathbf{e}^T\mathbf{q} = 0$ and the matrix

$$F + \mathbf{eq}^T = \frac{k}{k-1}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

has eigenvalues $0, -\frac{k}{k-1}, \ldots, -\frac{k}{k-1}$ and $\operatorname{rank}(F + \mathbf{eq}^T) = k - 1$. Since $m_a(0) = 1$, then from Lemma 2.2, F and $F + \mathbf{eq}^T$ are similar. Hence, $\operatorname{rank}(F) = \operatorname{rank}(F + \mathbf{eq}^T) = k - 1$. ∎

The following result allow us to construct doubly stochastic matrices having $(k-1)$ quadratic elementary divisors.

Theorem 2.1. *Let* $n, k \in \mathbb{Z}^+, n \geq 2k \geq 4$. *Then, there exists an* $n \times n$ *doubly stochastic matrix A with spectrum* $\Lambda = \{1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\}$, *and with elementary divisors*

$$(\lambda - 1), \underbrace{\left(\lambda + \frac{1}{n-1}\right)^2, \ldots, \left(\lambda + \frac{1}{n-1}\right)^2}_{(k-1)\text{-}times}, \underbrace{\left(\lambda + \frac{1}{n-1}\right), \ldots, \left(\lambda + \frac{1}{n-1}\right)}_{(n-2k+1)\text{-}times}.$$

Proof. Let F be the $k \times k$ matrix in (1) and let the $n \times n$ matrix

$$B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ F & & & 1 \end{bmatrix}.$$

It is clear that B is doubly quasi-stochastic, with all its eigenvalues equal to 1. Since $n \geq 2k,$ then $(B - I)^m = 0,\ m = 2, 3, \dots,$ and

$$r_1 = \operatorname{rank}(B - I) = \operatorname{rank}(F) = k - 1,$$

$$r_m = \operatorname{rank}(B - I)^m = 0, \quad m = 2, 3, \dots.$$

Thus, the number of Jordan blocks of B of sizes 1, 2, and 3, corresponding to the eigenvalue 1, is respectively,

$$d_1 - d_2 = r_0 - 2r_1 + r_2 = n - 2(k-1),$$

$$d_2 - d_3 = r_1 - 2r_2 + r_3 = k - 1,$$

$$d_3 - d_4 = 0.$$

Then,

$$\underbrace{(\lambda - 1)^2, \dots, (\lambda - 1)^2}_{(k-1)\text{-times}}, \underbrace{(\lambda - 1), \dots, (\lambda - 1)}_{n-2(k-1)\text{-times}}$$

are the elementary divisors of the doubly quasi-stochastic matrix B. From Lemma 2.1 with $\theta = -\frac{1}{n-1}$, and since $nJ_n - B$ is nonnegative,

$$A = \frac{1}{n-1}(nJ_n - B),$$

is doubly stochastic with spectrum Λ, and with the desired elementary divisors. ■

Lemma 2.4. *Let* $k \in \mathbb{Z}^+$ *with* $k \geq 2$. *Then,*

$$(\lambda - 1)^k, \underbrace{(\lambda - 1), \dots, (\lambda - 1)}_{k\text{-}times}, \tag{2}$$

are the elementary divisors of the $2k \times 2k$ *doubly quasi-stochastic matrix*

$$A_{2k} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ -1 & 1 & 1 & & & & & & \\ 1 & -1 & & 1 & & & & & \\ & & -1 & 1 & 1 & & & & \\ & & 1 & -1 & & \ddots & & & \\ & & & & \ddots & & 1 & & \\ & & & & & -1 & 1 & 1 & \\ & & & & & 1 & -1 & & 1 \end{bmatrix}. \quad (3)$$

Proof. Since $(A_{2k} - I)^m = 0$ for $m = k, k+1, \ldots,$ and

$$r_i = \operatorname{rank}(A_{2k} - I)^i = k - i, \quad i = 1, \ldots, k-1,$$

$$r_m = \operatorname{rank}(A_{2k} - I)^m = 0, \quad m = k, k+1, \ldots,$$

then the number of Jordan blocks of size 1 and k, corresponding to the eigenvalue 1, is respectively,

$$\begin{aligned} d_1 - d_2 &= r_0 - 2r_1 + r_2 \\ &= 2k - 2(k-1) + (k-2) \\ &= k, \end{aligned}$$

and

$$\begin{aligned} d_k - d_{k+1} &= r_{k-1} - 2r_k + r_{k+1} \\ &= k - (k-1) - 2(0) + 0 \\ &= 1. \end{aligned}$$

Thus, the result follows. ■

Theorem 2.2. *Let* $n, k \in \mathbb{Z}^+$, $n \geq 2k \geq 4$. *Then, there exists an* $n \times n$ *doubly stochastic matrix A, with eigenvalues* $1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}$, *and with elementary divisors*

$$(\lambda - 1), \left(\lambda + \frac{1}{n-1}\right)^k, \left(\lambda + \frac{1}{n-1}\right), \ldots, \left(\lambda + \frac{1}{n-1}\right).$$

Proof. Let A_{2k} be the matrix in (3). Then, the $n \times n$ matrix

$$B = \begin{bmatrix} A_{2k} & \\ & I_{n-2k} \end{bmatrix},$$

is doubly quasi-stochastic with all its eigenvalues equal to 1. From Lemma 2.4, B has elementary divisors

$$(\lambda - 1)^k, \underbrace{(\lambda - 1), \ldots, (\lambda - 1)}_{(n-k)\text{- times}}.$$

Then, from Lemma 2.1, and since $nJ_n - B$ is nonnegative,

$$A = \frac{1}{n-1}(nJ_n - B),$$

is doubly stochastic with spectrum Λ and with the desired elementary divisors. ∎

Theorem 2.3. *Let* $n, k \in \mathbb{Z}^+, k \geq 2, n \geq k(k+1) - 2.$ *Then, there exists an* $n \times n$ *doubly stochastic matrix A with eigenvalues* $1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}$, *and with elementary divisors*

$$(\lambda - 1), \underbrace{\left(\lambda + \frac{1}{n-1}\right), \ldots, \left(\lambda + \frac{1}{n-1}\right)}_{\left\{n - \frac{k(k+1)}{2}\right\}\text{- times}},$$

$$\left(\lambda + \frac{1}{n-1}\right)^2, \left(\lambda + \frac{1}{n-1}\right)^3, \ldots, \left(\lambda + \frac{1}{n-1}\right)^{k-1}, \left(\lambda + \frac{1}{n-1}\right)^k.$$

Proof. Let $A_{2i}, i = 2, 3, \ldots, k,$ be the $2i \times 2i$ matrix in (3) and let the $n \times n$ matrix

$$B = \begin{bmatrix} A_4 & & & & & \\ & A_6 & & & & \\ & & A_8 & & & \\ & & & \ddots & & \\ & & & & A_{2k} & \\ & & & & & I_{n-k(k+1)+2} \end{bmatrix}.$$

From Lemma 2.4, the matrix B has elementary divisors

$$\underbrace{(\lambda - 1), \ldots, (\lambda - 1)}_{\left\{n - \frac{k(k+1)}{2} + 1\right\} \text{veces}}, (\lambda - 1)^2, (\lambda - 1)^3, (\lambda - 1)^4, \ldots, (\lambda - 1)^k.$$

Then, from Lemma 2.1, and since $nJ_n - B$ is nonnegative,

$$A = \frac{1}{n-1}(nJ_n - B),$$

is doubly stochastic with spectrum Λ and with the desired elementary divisors. ■

3. Examples

Example 3.1. From Theorem 2.1,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & 1 \end{bmatrix},$$

is doubly quasi-stochastic and

$$A = \frac{1}{5}(6J_6 - B) = \frac{1}{5}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 1 \\ \frac{1}{2} & 2 & \frac{1}{2} & 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 2 & 1 & 1 & 0 \end{bmatrix},$$

is doubly stochastic with elementary divisors

$$(\lambda - 1), \left(\lambda + \frac{1}{5}\right)^2, \left(\lambda + \frac{1}{5}\right)^2, \left(\lambda + \frac{1}{5}\right).$$

Example 3.2. From Theorem 2.3,

$$A = \frac{1}{9}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

has elementary divisors

$$(\lambda - 1), \left(\lambda + \frac{1}{n-1}\right)^3, \left(\lambda + \frac{1}{n-1}\right)^2, \left(\lambda + \frac{1}{9}\right), \ldots, \left(\lambda + \frac{1}{9}\right).$$

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